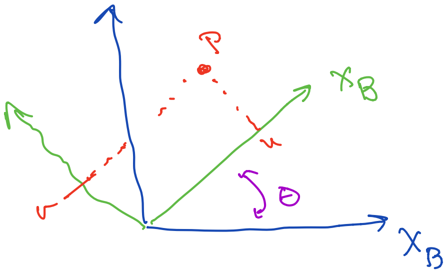
Review: SO(n) is a Lie Group.

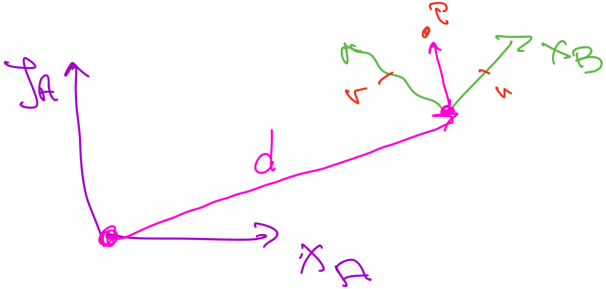
Coordinate (rotational) transformations.



$P^B = \begin{bmatrix} u \\ v \end{bmatrix}$

$\Rightarrow \quad P^A = R^A_B \, P^B$

$R^A_B = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \in SO(2)$



$P^A = \underbrace{R^A_B P^B}_{\text{As Before}} + \underbrace{d^A_B}_{\text{from origin of A to origin of B}}$

R, d tell me everything about orientation & position of one frame w.r.t. another.

$\Rightarrow$ $SE(n) = SO(n) \times \mathbb{R}^n$ [modulo some nuance]

↳ Special Euclidean Group of Order n

$\Rightarrow$ $SE(n)$ is a Lie Group!

An element of $SE(n)$ can be expressed as a homogeneous transformation matrix of the form

$$\left[\begin{array}{ccc|c} & R & & d \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] = T$$

Coord. Transformations using homogeneous Transformation matrices:

$$\left.\begin{aligned} p^A &= R^A_B\, p^B + d^A_B \\ 1 &= 0 + 1 \end{aligned}\right\} \quad \begin{bmatrix} p^A \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} R^A_B & d^A_B \\ \underline{0} & 1 \end{array}\right]}_{T^A_B} \begin{bmatrix} p^B \\ 1 \end{bmatrix}$$

Some people write this as

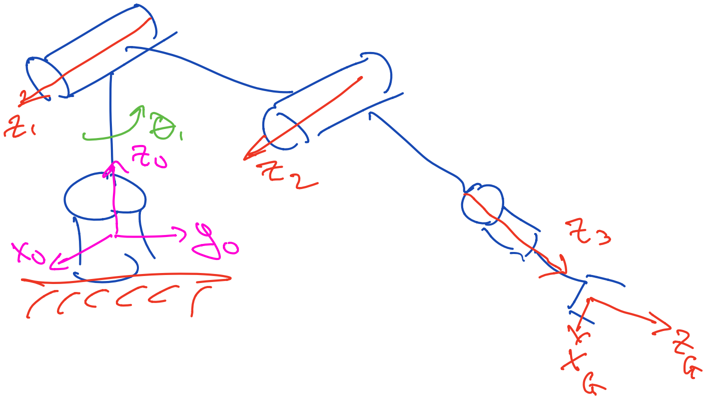
$$\tilde{p}^A = T^A_B\, \tilde{p}^B \quad \text{where} \quad \tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}$$

↳ Homogeneous coordinates

Robot Arm Kinematics

Where is frame G w.r.t. a ref frame, say frame 0.

⟹ Frame 0 has Z_0 axis as axis of revolution for 1st joint.



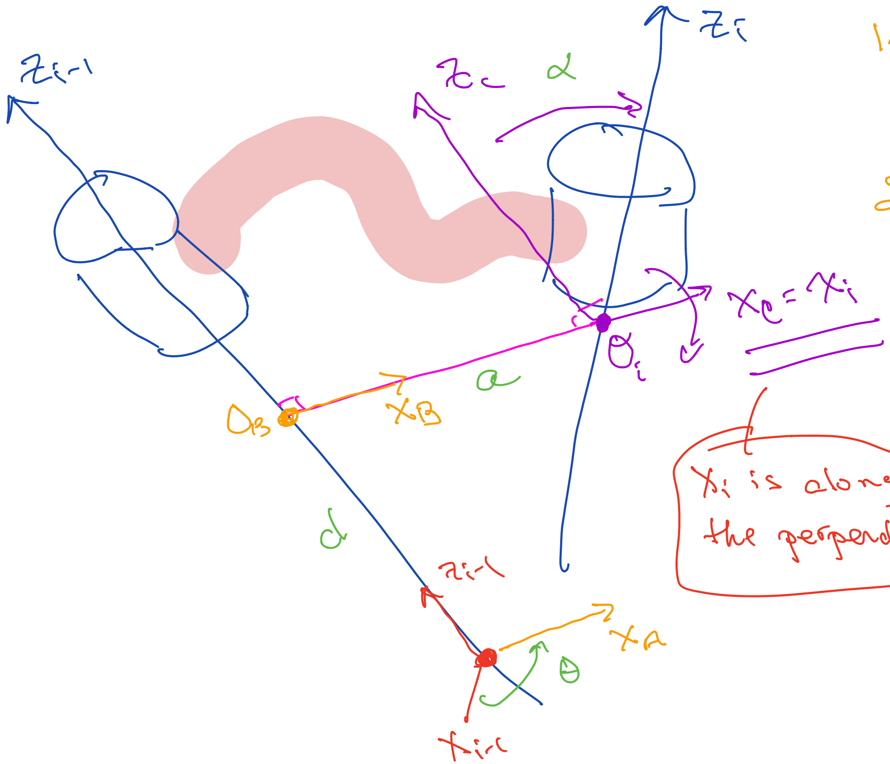
⟹ To answer my question, need to compute T^0_G

Now $T^0_G = T^0_1 \, T^1_2 \, T^2_3 \, T^3_G$

$= T^0_1(\theta_1) T^1_2(\theta_2) T^2_3(\theta_3) T^3_G(\theta_4)$

Map from $\theta_1, \cdots \theta_n \longrightarrow T^0_G$

is called Forward Kinematic map



x_i is along the perpendicular

1. rotate about z_{i-1} to align x_A w/ perpendicular
2. translate along z_{i-1} s.t. origin O_B is at the perpendicular
3. translate along x_B to z_i, z_c still parallel to z_{i-1}
4. rotate about x_c to align z axes

$$T_i^{i-1} = \underbrace{\begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\theta} \underbrace{\begin{bmatrix} I & \begin{matrix} 0 \\ 0 \\ d \end{matrix} \\ 0\ 0\ 0 & 1 \end{bmatrix}}_{d} \underbrace{\begin{bmatrix} I & \begin{matrix} a \\ 0 \\ 0 \end{matrix} \\ 0\ 0\ 0 & 1 \end{bmatrix}}_{a} \underbrace{\begin{bmatrix} c\alpha & -s\alpha & 0 & 0 \\ s\alpha & c\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\alpha}$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{constant} \end{bmatrix} \in SE(3)$$

$\Rightarrow T_i^{i-1}(\theta_i)$ $\boxed{\text{param'd by } \alpha_i, a_i, d_i}$ — constants

Denavit-Hartenberg (DH)

Summary

$$T^0_n(q) = T^0_1(\theta_1) \cdots T^{n-1}_n(\theta_n) \Leftarrow \text{Forward Kinematics}$$

$\hookrightarrow$ configuration $q = (\theta_1, \ldots \theta_n)$

$$T^0_n : Q \longrightarrow \underbrace{SE(3)}$$

Q ↑ conf. Space

$SE(3)$ ↑ Task Space

what about 'inverse kinematics'??

$\longrightarrow$ later

Differential Kinematics

we know $T: \Theta \to SE(3)$ $\rightsquigarrow$ $\theta_1, \cdots \theta_n$ fixed

T tells where is gripper frame

what about $\frac{d}{dt} T$??

$\rightarrow$ $\theta_1, \cdots \theta_n$ are changing

What is the gripper motion?

$\rightarrow$ Relationship between $\dot{\theta}_1, \dot{\theta}_2 \cdots, \dot{\theta}_n$ and "velocity" of gripper frame.

$$T(t) = \begin{bmatrix} R(t) & d(t) \\ 0 & 1 \end{bmatrix} \Longrightarrow \frac{d}{dt} T(t) = \begin{bmatrix} \frac{d}{dt} R & \frac{d}{dt} d \\ 0 & 0 \end{bmatrix}$$

$\frac{d}{dt} d = \underline{\underline{\text{easy}}} \Rightarrow$ linear velocity of origin of moving frame

what about $\frac{d}{dt} R$?? $\Longrightarrow$ $SO(n)$ a lie group, so we can solve this

$SO(n)$ a lie group, its tangents live in $so(n)$ a Lie Algebra

lower case

New Concept: Skew Symmetric Matrix

Def If S is a skew sym. matrix then $S + S^T = 0$.

Def The set of $n \times n$ skew sym. Matrices is called $so(n)$. (lower case)

The matrices in $so(n)$ have some properties:

- For $so(3)$, $S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$ $\quad a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$

  Sk. sym. matrix operator

  aka: $a\times$, $\hat{a}$

- $S(a)b = a \times b$

  Vector cross product

- $R(a \times b) = (Ra) \times (Rb)$ for $R \in SO(n)$, $a, b \in \mathbb{R}^n$

- $R\, S(a) R^T = S(Ra)$ [handy for $\frac{d}{dt}(R_1 R_2 \cdots R_n)$]

Differentiation of $R \in SO(n)$

$$RR^T = I \Rightarrow \frac{d}{dt}(RR^T) = 0$$

$$\frac{d}{dt}(RR^T) = \underbrace{\underbrace{\dot{R}R^T}_{S} + \underbrace{R\dot{R}^T}_{S^T} = 0}_{\dot{R}R^T \in so(n)\ !!!}$$

Notice

$$[\dot{R}R^T]^T = R\dot{R}^T$$

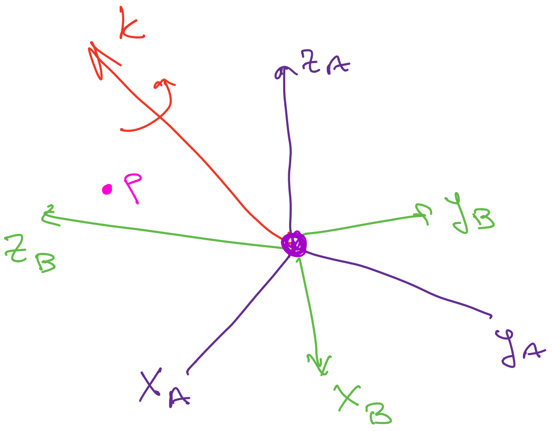
(cont)

$$\dot{R}R^T = S(\omega)$$

$$\boxed{\dot{R} = S(\omega)R}$$

ω is some n-vector that parameterizes the sk. sym matrix $S(\omega)$

Relationship to Familiar Ideas



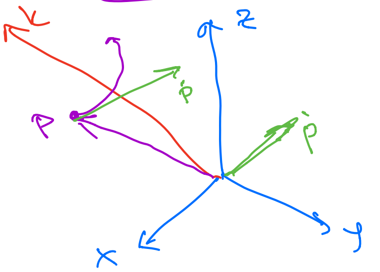
- Frame B is rotating about axis k at speed $\dot{\theta}$.
- P is rigidly attached to frame B.

From Freshman Physics: $\boxed{v = \omega \times r}$

$$\dot{p} = \dot{\theta} k \times p$$

w.r.t. Frame A

$$\dot{p}^A = \dot{\theta} k^A \times p^A$$



$$\dot{p}^A = \dot{\theta} k^A \times p^A$$

$$= \dot{\theta} k^A \times R^A_B p^B$$

$$= \boxed{S(\dot{\theta} k^A) R^A_B p^B}$$

From the $\frac{d}{dt} R$ perspective

$$p^A = R^A_B p^B$$

$$\frac{d}{dt} p^A = \frac{d}{dt} R^A_B p^B \qquad (\text{recall } \dot{p}^B = 0)$$

$$= \dot{R}^A_B p^B$$

$$= \boxed{S(\omega) R^A_B p^B}$$

$$\boxed{S(\dot{\theta} k^A) R^A_B p^B = S(\omega) R^A_B p^B}$$

$$\boxed{\dot{\theta} k^A = \omega}$$

Notation

$$\omega^A_{\underbrace{A,B}_{R^A_B}}$$

Express ω w.r.t. Frame A

Easy Extension

linear velocity of a point attached to a moving frame

$$p^A = \underbrace{R^A_B p^B}\, + d^A_B \qquad \text{i.e., } T^A_B = \left[\begin{array}{c|c} R^A_B & d^A_B \\ \hline 0 & 1 \end{array}\right]$$

$$\dot{p}^A = S(\omega^A_{A,B}) R^A_B p^B + \dot{d}^A_B$$

$$\left[\begin{array}{c} \dot{p}^A \\ 0 \end{array}\right] = \left[\begin{array}{c|c} S(\omega^A_{A,B}) R^A_B & \dot{d}^A_B \\ \hline 0 \quad 0 \quad 0 & 0 \end{array}\right] \left[\begin{array}{c} p^B \\ 1 \end{array}\right]$$

$\rightarrow \in se(3)$

Lie Algebra

Back to Robots

$$T_n^0 = T_1^0 \cdots T_n^{n-1} = \left[\begin{array}{c|c} R_1^0 & \\ \hline & \end{array}\right]\left[\begin{array}{c|c} R_2^1 & \\ \hline & \end{array}\right] \cdots \left[\begin{array}{c|c} R_n^{n-1} & \\ \hline & \end{array}\right]$$

not complicated

complicated

$$= \left[\begin{array}{c|c} R_1^0 R_2^1 \cdots R_n^{n-1} & \{\ \} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

$$\Rightarrow \boxed{\frac{d}{dt} R_n^0 = \frac{d}{dt}\left(R_1^0 R_2^1 \cdots R_n^{n-1}\right)}$$

Addition of Angular Velocities

$$R_2^0 = R_1^0 R_2^1 \implies \boxed{\dot{R}_2^0 = S(\omega_{0,2}^0) R_2^0}$$

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1 \quad \text{(prod rule)}$$

$$\boxed{S(\omega_{0,1}^0) R_1^0 R_2^1} + R_1^0 S(\omega_{1,2}^1) R_2^1$$

$$\boxed{R^T R = I}$$

$$R_2^0 \qquad R_1^0 S(\omega_{1,2}^1)(R_1^0)^T R_1^0 R_2^1$$

$$R_1^0 S(\omega_{1,2}^1) R_1^{0T} R_2^0$$

$$\boxed{R_1^0 \omega_{1,2}^1 = \omega_{1,2}^0}$$

$$\boxed{S(R_1^0 \omega_{1,2}^1) R_2^0}$$

$$\implies S(\omega_{0,2}^0) R_2^0 = S(\omega_{0,1}^0) R_2^0 + S(\omega_{1,2}^0) R_2^0$$

$$\Rightarrow \boxed{\omega^0_{0,2} = \omega^0_{0,1} + \omega^0_{1,2}}$$

In general

$$\frac{d}{dt} R^0_n \Rightarrow \boxed{\omega^0_{0,n} = \sum_{i=1}^{n} R^0_{i-1}(t)\, \omega^{i-1}_{i-1,i}(t)}$$

$$= \sum_{i=1}^{n} \underbrace{\omega^0_{i-1,i}}$$

Jacobian (Manipulator)

$$\begin{bmatrix} V_n^0 \\ W_{0,n}^0 \end{bmatrix} = \underbrace{J(q)}_{\text{Jacobian}}\ \dot{q}$$

$\dot{q}$ ← joint velocity vector

$\begin{bmatrix} \text{linear vel} \\ \text{angular vel} \end{bmatrix}$

$$W_{0,n}^0 = \sum R_{i-1}^0\, W_{i-1,i}^{i-1}$$

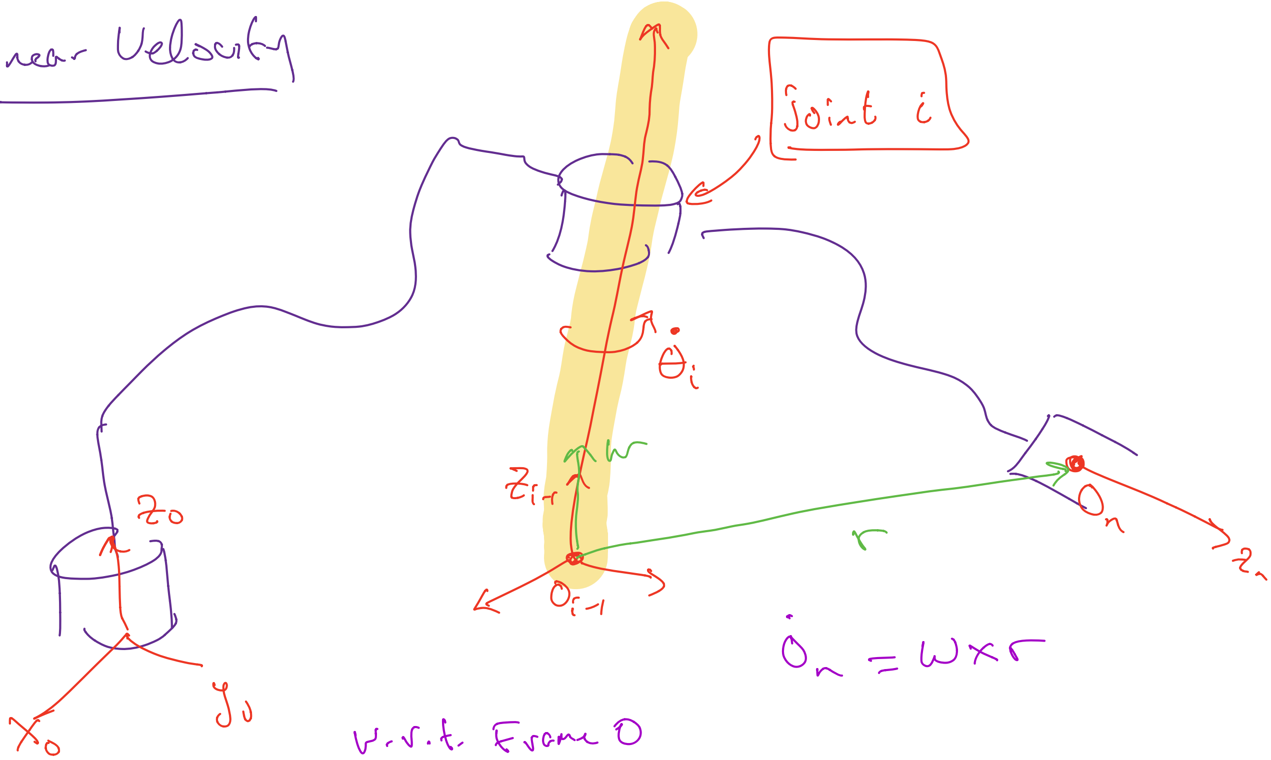
Angular velocity

$$\left.\underbrace{R_{i-1}^0}_{\boxed{R_1^0 R_2^1 \cdots R_{i-1}^{i-2}}} W_{i-1,i}^{i-1}\right\} \Longrightarrow \boxed{Z_{i-1}^0\, \dot{\theta}_i}$$

w

$$W_{i-1,i}^{i-1} \Longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_i = Z_{i-1}^{i-1}\, \dot{\theta}_i$$

Linear Velocity



$$\dot{O}_n = \omega \times r$$

w.r.t. Frame 0

$$\dot{O}_n^0 = \left(\dot{\theta}_i \, z_{i-1}^0\right) \times \left(O_n^0 - O_{i-1}^0\right)$$

$$J = [J_1 \; J_2 \cdots J_n]$$

$$\underbrace{J_i = \begin{bmatrix} z_{i-1}^0 \times (O_n^0 - O_{i-1}^0) \\ z_{i-1}^0 \end{bmatrix}}_{\boxed{\text{For revolute joint } i}}$$

$J_i \dot{\theta}_i = \begin{bmatrix} v \\ \omega \end{bmatrix}$ if only $\dot{\theta}_i \neq 0$

$$\omega_{i-1,i}^{i-1} \Longleftrightarrow \underbrace{\frac{d}{dt} R_i^{i-1}}_{\text{angular vel about } z_{i-1}}$$